# Lecture Notes 10

## 1. Stationary – fancy word

***1. Stationary – time series does not depend on time.***

Does not vary with time

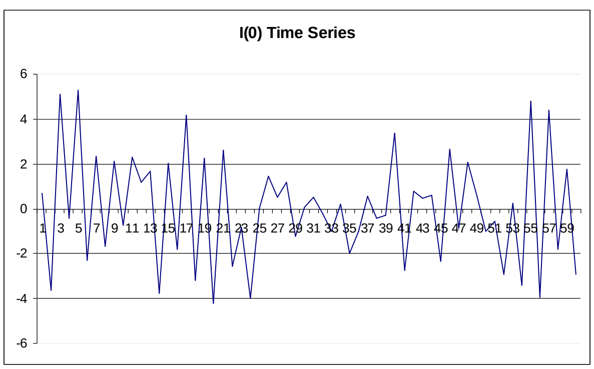
Weakly stationary – the mean and variance of a time series does not vary with time

Examples

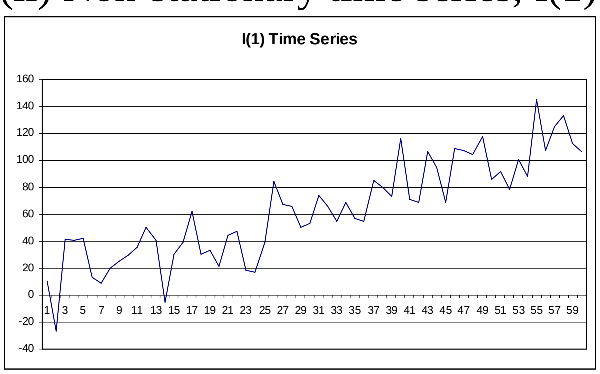
“I” means integrative

Integration is the branch of calculus that deals with infinite sums, like area under a function.

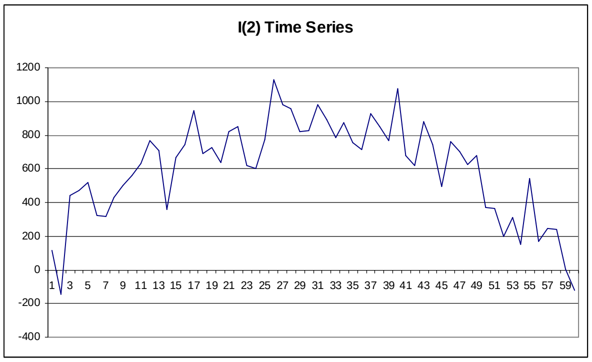
(i) Stationary time series, I(0) – no trend




(ii) Non-stationary time series, I(1) – linear trend



(ii) Non-stationary time series, I(2) – Quadratic trend




***2. MA(q) Process***

Moving average has q lags

$$X_t = Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2} + \cdots + \theta_q Z_{t-q}$$

$Z_t$ ~ white noise
$Z_t \sim (0, \sigma^2)$

Since $X_t$ is a function of white noise, then it is stationary. Series does not move in any direction.

***3. AR(1) Process***

$$X_t = \phi X_{t-1} + Z_t$$

$Z_t$ ~ white noise
Very important $|\phi| < 1$

Note – AR(1) can be written as $X_t - \phi X_{t-1} = Z_t$

We will do a trick, here is the AR(1)

$$X_t = \phi X_{t-1} + Z_t$$

If we go back one unit in time, then

$$X_{t-1} = \phi X_{t-2} + Z_{t-1}$$

Substitute $X_{t-1}$ into the $X_t$ equation to get:

$$X_t = \phi(\phi X_{t-2} + Z_{t-1}) + Z_t$$

$$X_t = \phi^2 X_{t-2} + Z_t + \phi Z_{t-1}$$

Let's go back another unit in time, so

$$X_{t-2} = \phi X_{t-3} + Z_{t-2}$$

Substitute this into $X_t$ equation, yielding

$$X_t = \phi^2(\phi X_{t-3} + Z_{t-2}) + Z_t + \phi Z_{t-1}$$

$$X_t = \phi^3 X_{t-3} + Z_t + \phi Z_{t-1} + \phi^2 Z_{t-2}$$

We keep doing this trick until we get the infinite series:

$$X_t = Z_t + \phi Z_{t-1} + \phi^2 Z_{t-2} + \phi^3 Z_{t-3} \cdots$$

Thus, $X_t$ as an AR(1) process can be written as an infinite Moving Average time series.

Did you notice the $\phi$'s; the further back in time you go, the smaller the $\phi$'s are because they are raised to a power. Since $\phi$'s are fractions, they get smaller when raised to higher and higher powers.

Thus, the AR(1) is a stationary process.

You can show an AR(p) is stationary too in the same manner. It just involves more algebra.

Note – The Random Walk is <u>not</u> a stationary process

$$X_t = X_{t-1} + Z_t$$

$Z_t$ ~ white noise

Note $\phi = 1$

Do the same thing, the time series for the last time period is:

$$X_{t-1} = X_{t-2} + Z_{t-1}$$

Substitute $X_{t-1}$ into $X_t$ equation, yielding:

$$X_t = X_{t-2} + Z_t + Z_{t-1}$$

The time series for two periods ago is:

$$X_{t-2} = X_{t-3} + Z_{t-2}$$

Substitute this into $X_t$ and keep going.

$$X_t = X_{t-3} + Z_t + Z_{t-1} + Z_{t-3}$$

Keep doing this until $X_t$ is written as an infinite series,

$$X_t = Z_t + Z_{t-1} + Z_{t-3} + Z_{t-4} + \cdots$$

A random walk is not stationary. Why? Any noise from the past gets added at full value at time t. Thus, the further in time you go, the more noise is added to the time series. That is why a random walk tends to move in one direction, even though a random walk is composed of random numbers.

### *4. Backwards Shift Operator (B)*

AR(1) Process

$$X_t - \phi X_{t-1} = Z_t$$

$Z_t$ ~ white noise

$Z_t \sim (0, \sigma^2)$

Very important $|\phi| < 1$

"B" is backwards shift operator;

A function that is treated as a variable

AR(1) can be written as,

$$(1 - \phi B)X_t = Z_t$$

Let's show this is the same thing by using algebra,

$$(1 - \phi B)X_t = Z_t$$
$$X_t - \phi B X_t = Z_t$$
$$X_t - \phi X_{t-1} = Z_t$$

Using the backwards shift operator, let's show the AR(1) written as an infinite MA process,

$$(1-\phi B)X_t = Z_t$$

$$X_t = \frac{1}{1-\phi B} Z_t$$

Since $|\phi| < 1$, then this is written as an infinite series,

$$X_t = \left(1 + \phi B + \phi^2 B^2 + \phi^3 B^3 + \cdots\right) Z_t$$

$$X_t = Z_t + \phi B Z_t + \phi^2 B^2 Z_t + \phi^3 B^3 Z_t + \cdots$$

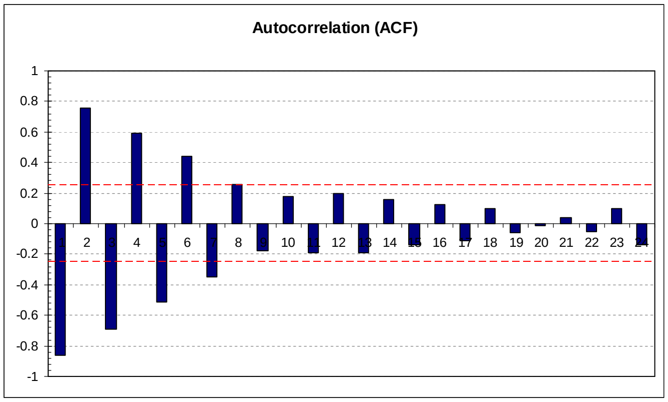
$$X_t = Z_t + \phi Z_{t-1} + \phi^2 Z_{t-2} + \phi^3 Z_{t-3} + \cdots$$

What does this mean?

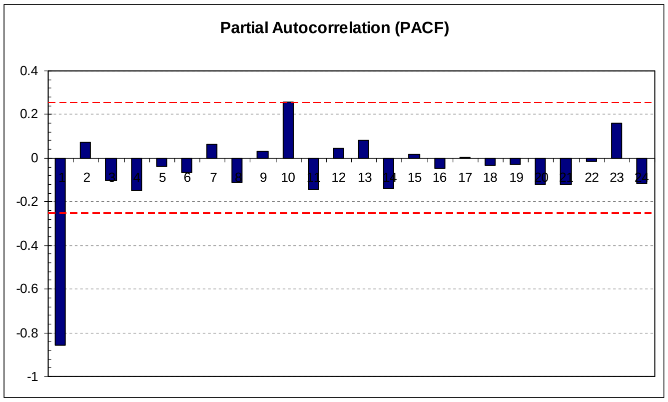
An AR(1) will have one statistically significant lag, h=1, on the PACF and the ACF will tail off.

If the AR(1) is negative autocorrelation ($\phi<0$), then the ACF and PACF plots will look like,

ACF is "tailing off"

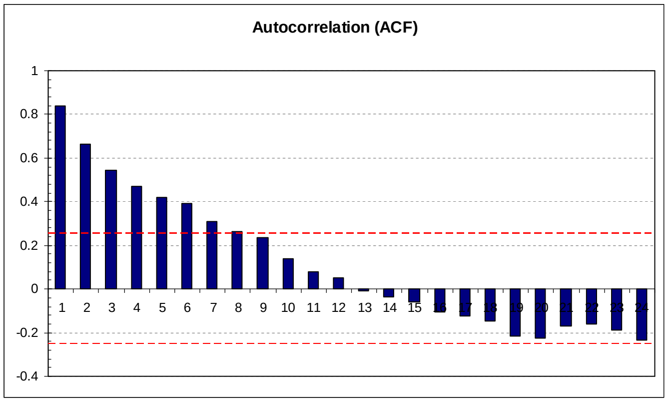



PACF has a negative significant lag at h=1

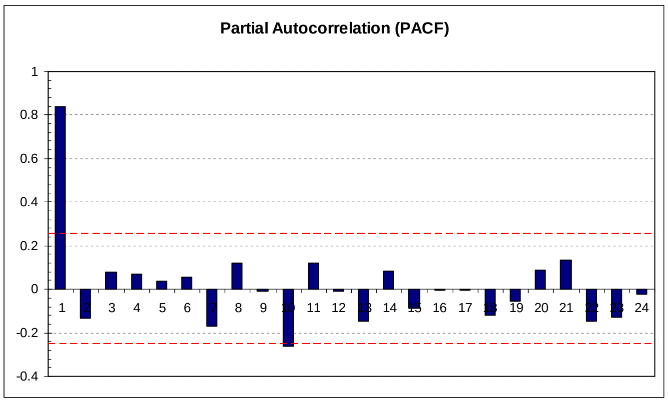


If the AR(1) is positive autocorrelation ($\phi > 0$), then the ACF and PACF plots will look like,

ACF plot "tails off"




PACF Plot has one significant lag at h=1




Note – the 95% confidence intervals are the red dashed lines. Thus, 5% of the time, a lag may be statistically significant due to random chance. For instance, lag 10 on the PACF plot is significant, but should be ignored.

MA(1) Process

$$X_t = Z_t + \theta Z_{t-1}$$

$Z_t$ ~ white noise

$Z_t \sim (0, \sigma^2)$

Very important $|\theta| < 1$

Writing the MA(1) using backshift operator, then:

$$X_t = (1 + \theta B)Z_t$$

Using algebra to get original equation:

$$X_t = (1 + \theta B)Z_t$$
$$X_t = Z_t + \theta B Z_t$$
$$X_t = Z_t + \theta Z_{t-1}$$

Use algebra to show that an MA(1) process can be written as an infinite series of AR process.

$$X_t = (1 + \theta B)Z_t$$
$$\frac{1}{1 + \theta B} X_t = Z_t$$

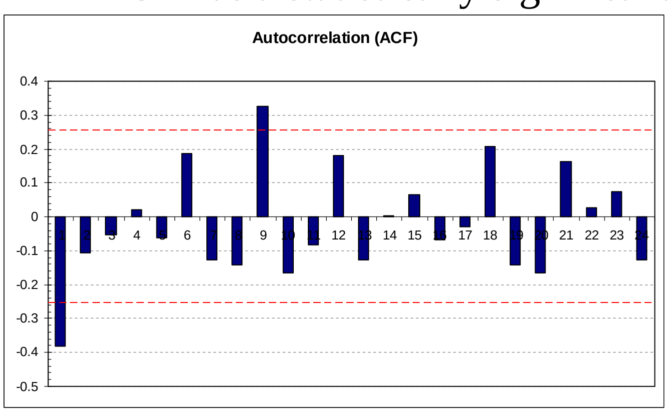
Since $|\theta| < 1$, then the fraction can be expanded into an infinite series,

$$(1 + \theta B + \theta^2 B^2 + \theta^3 B^3 + \cdots)X_t = Z_t$$
$$X_t - \theta B X_t - \theta^2 B^2 X_t - \theta^3 B^3 X_t + \cdots = Z_t$$
$$X_t - \theta X_{t-1} - \theta^2 X_{t-2} - \theta^3 X_{t-3} + \cdots = Z_t$$

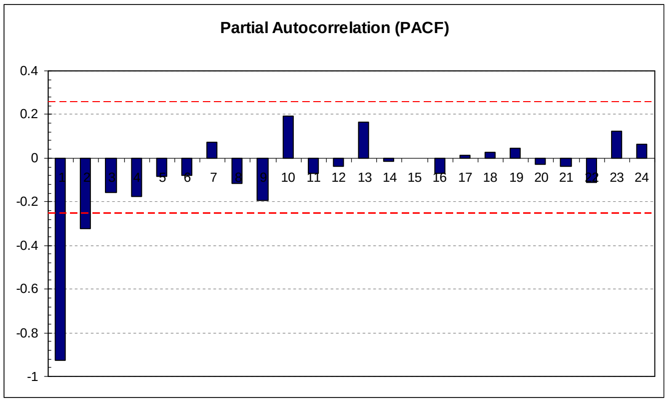
What does this mean? An MA(1) process will have a statistically significant lag for h=1 on the ACF plot and the PACF plot will trail off

If the MA(1) has a negative theta ($\theta < 0$), then the ACF and PACF plots will look like,

ACF has a statistically significant lag, h=1

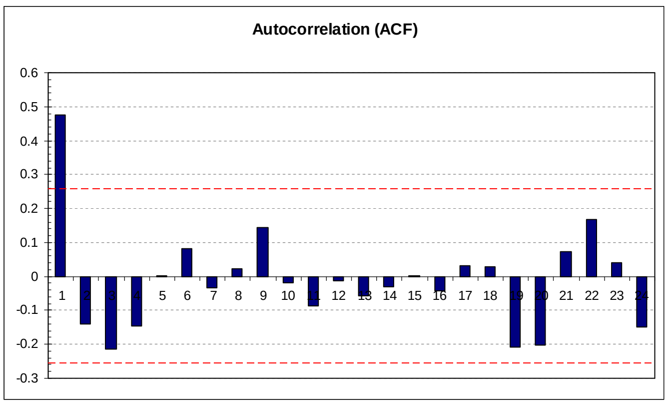



PACF should "trail off" This is not a good example of trailing off.

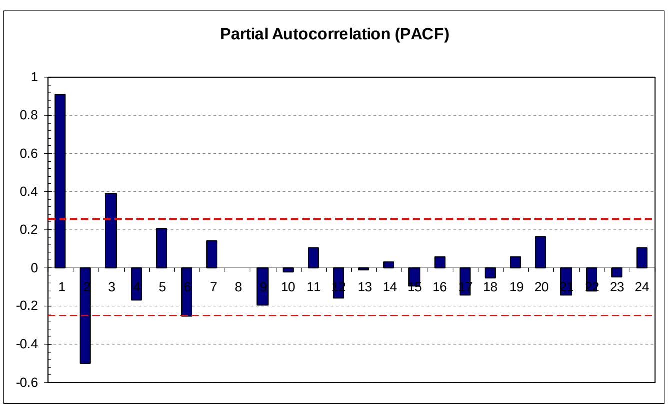



If the MA(1) has a positive theta ($\theta > 0$), then the ACF and PACF plots will look like the following:

ACF has a statistically significant lag, h=1




PACF should “trail off.”




ARMA(1,1) Process

$$X_t - \phi X_{t-1} = Z_t + \theta Z_{t-1}$$

$Z_t$ ~ white noise
$Z_t \sim (0, \sigma^2)$
Very important $|\phi| < 1$
and $\phi + \theta \neq 0$

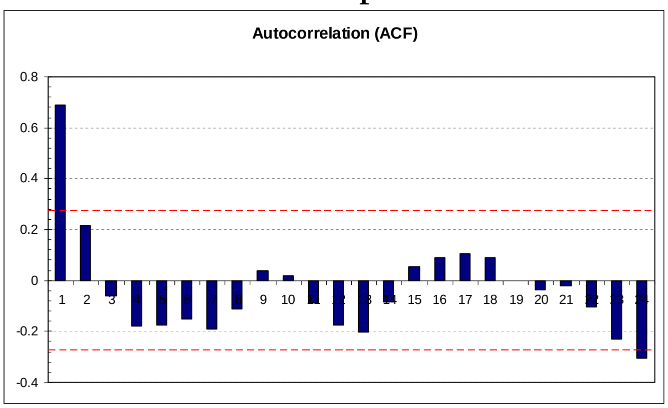
Rewrite in Backwards shift operator, then

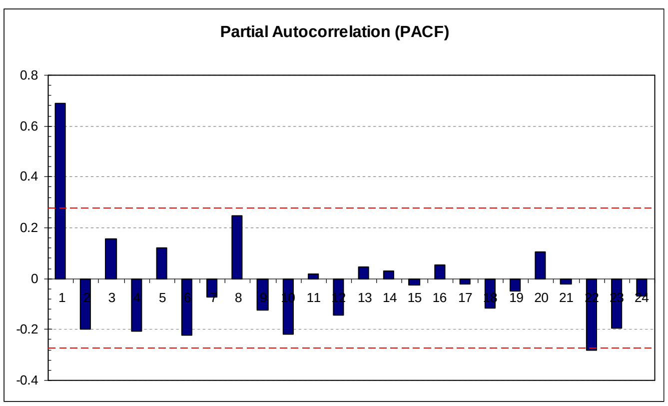
$$(1-\phi B)X_t = (1+\theta B)Z_t$$

Using algebra to show this is a normal ARMA(1,1)

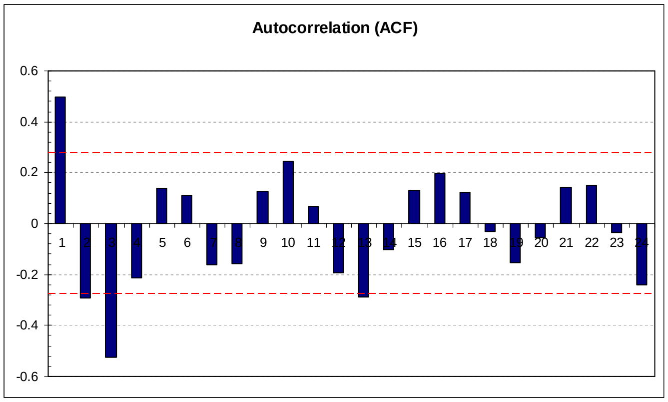
$$(1-\phi B)X_t = (1+\theta B)Z_t$$
$$X_t - \phi B X_t = Z_t + \theta B Z_t$$
$$X_t - \phi X_{t-1} = Z_t + \theta Z_{t-1}$$

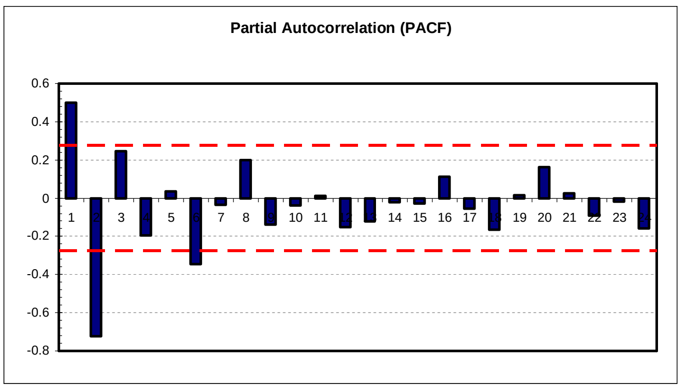
The ACF and PACF plots look like the following:




Look at the example below and predict which ARMA to estimate this model with:




Both the ACF and PACF do not appear to tail off. Thus the ACF has three significant lags and the PACF has two significant lags. Thus, I would estimate an ARMA(2,3). The 2 is for the autoregressive and 3 is for the moving average terms.

Then we use the ARMA estimator to estimate a model for this choice.

Note – I have notice that these procedures work well for autoregressive terms. The moving average terms tend to be more difficult.

***5. First Differences***

Take data and perform a first difference. This converts an I(1) time series into an I(0) time series.

First Example

| $X_t$ | First difference |
|---|---|
| 100 | NA |

| | |
|---|---|
| 110 | =110 – 100 = 10 |
| 125 | =125 – 110 = 15 |
| 136 | =136 – 125 = 11 |

Second Example

You have a simple linear trend regression, $X_t = \beta_1 + \beta_2 t + u_t$

The linear trend is not stationary; it depends on time.

The trend regression is the following for the last time period,

$$X_{t-1} = \beta_1 + \beta_2(t-1) + u_{t-1}$$

Take the first difference, $X_t - X_{t-1}$, which is

$$X_t - X_{t-1} = \beta_1 + \beta_2 t + u_t - (\beta_1 + \beta_2(t-1) + u_{t-1})$$

$$X_t - X_{t-1} = \beta_1 + \beta_2 t + u_t - \beta_1 - \beta_2(t-1) - u_{t-1}$$

$$X_t - X_{t-1} = \beta_2 + u_t - u_{t-1}$$

The first difference creates a time series that is stationary some mean, $\beta_2$.

Although we have two error terms, this is not a random walk.

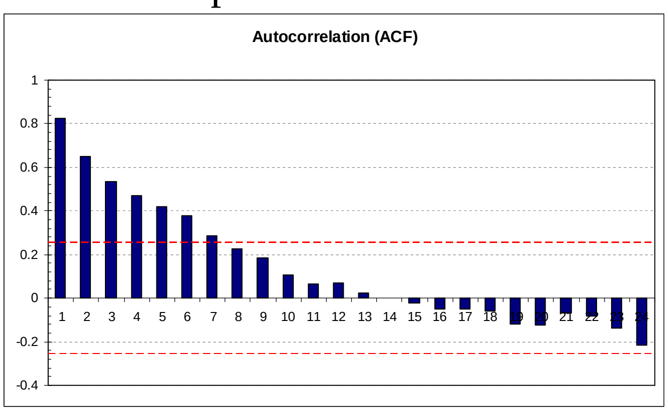
Third Example – Random Walk

$$X_t = X_{t-1} + Z_t$$

$Z_t$ ~ white noise

Note $\phi = 1$

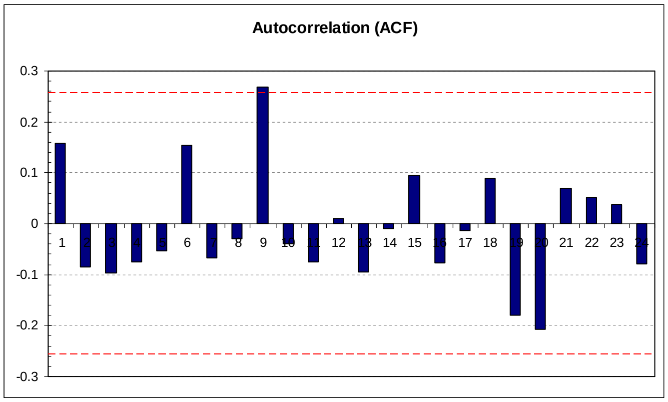
The random walk is not stationary. All non-stationary time series has an ACF plot that "tails off" like the one below:

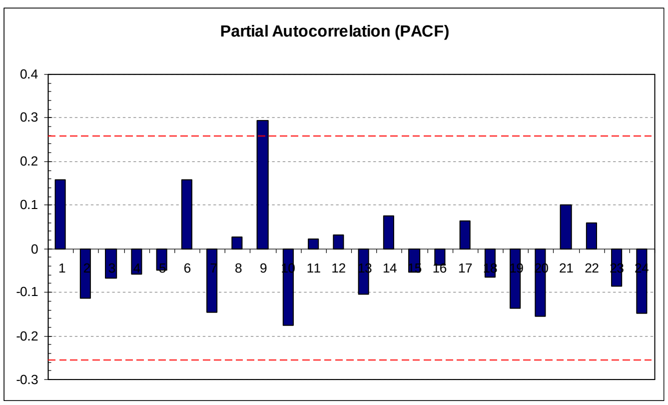


If we do a first different of a random walk, then

$$X_t - X_{t-1} = Z_t$$

After the first difference, the time series should be noise. The ACF and PACF plots look like:




A time series that is pure noise should not have any statistically significant lags. In our case, one lag is significant, which may be due to chance.

Most common time series

| | | |
|---|---|---|
| I(0) | Stationary | Very common |
| I(1) | Linear trend | Very common |
| I(2) | Quadratic trend | Not so common, but it does arise |

U.S. GDP to debt ratio
U.S. Petroleum production

Second difference – converts the I(2) time series into a stationary time series, I(0).

| $X_t$ | First Difference | Second Difference |
|---|---|---|
| 100 | NA | NA |
| 110 | 10 | NA |
| 115 | 5 | = 5 – 10 = -5 |
| 125 | 10 | = 10 – 5 = 5 |
| 135 | 10 | = 10 – 10 = 0 |

Why is this important?

1. A time series has to be stationary in order to estimate an ARMA(p,q), using the Excel program.
2. If you use linear regression,

   $$Y_t = \beta_1 + \beta_2 X_{2t} + \cdots \beta_k X_{kt} + u_t$$

   The residuals can be estimated as an ARMA process, thus

   $u_t$ ~ ARMA(p, q) process

   Total number of parameters = k + p + q

   Simple ARMAs can be estimated in Excel
3. Spurious regression

   $$Y_t = \beta_1 + \beta_2 X_{2t} + \cdots \beta_k X_{kt} + u_t$$

   The Yt and all X's have to have the same order of integration.

   All variables are I(0)

   Or all variables are I(1)

   You cannot mix and match the I(0) and I(1), otherwise you have spurious regression.

   Example

   $$\underset{I(1)}{Y_t} = \beta_1 + \beta_2 \underset{I(0)}{X_{2t}} + u_t$$

   You can get weird results

   $Y_t$ is the tea price in China, I(1)

   $X_{2t}$ is sunspot activity, I(0)

Logic would dictate that these two variables are not related. However, regression may find a relationship, because you mixed orders of integration.